S. -T. Yau College Student Mathematics Contest
Applied and Computational Math (Individual Overall Contest)

June 8, 2024

1. Consider the following optimization problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \ \|\boldsymbol{x}\|_1 \text{ s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}_0, \tag{1}$$

where $\|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m \le n$, $\boldsymbol{x}_0 \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. Let $s < n$ be an integer. Define the set

$$\mathcal{C}_s = \{\boldsymbol{\eta} \in \mathbb{R}^n \mid \exists\, S \subset \{1,2,\ldots,n\}, \text{ s.t. } |S| \le s \text{ and } \|\boldsymbol{\eta}_S\|_1 \ge \|\boldsymbol{\eta}_{S^c}\|_1,\}.$$

Here $\boldsymbol{\eta}_S \in \mathbb{R}^n$ is the vector whose components are the same as $\boldsymbol{\eta}$ on S and 0 on $S^c := \{1,\ldots,n\} \setminus S$.

   (a) Prove that every s-sparse vector $\boldsymbol{x}_0$ is the unique solution of (1) if and only if

   $$\mathrm{Ker}(\boldsymbol{A}) \cap \mathcal{C}_s = \{\boldsymbol{0}\}. \tag{2}$$

   (A vector is called s-sparse if it contains at most s nonzero entries.)

   (b) Prove that there exists a universal constant $C > 0$ such that

   $$\mathcal{C}_s \cap \{\boldsymbol{x} \mid \|\boldsymbol{x}\|_2 \le 1\} \subseteq \{\boldsymbol{x} \mid \|\boldsymbol{x}\|_s \le C\},$$

   where $\|\boldsymbol{x}\|_{(s)}$ is defined as

   $$\|\boldsymbol{x}\|_{(s)} := \min\left\{ \sum_i \|\boldsymbol{x}_i\|_2 \mid \boldsymbol{x} = \sum_i \boldsymbol{x}_i, \text{ where } \boldsymbol{x}_i \in \mathbb{R}^n \text{ is } s\text{-sparse for all } i. \right\}$$

   (Since in the general case $\|\boldsymbol{x}\|_{(s)} \le \sqrt{n/s}\|\boldsymbol{x}\|_2$, the above result implies that $\mathcal{C}_s$ restricted to the 2-norm ball is very small, indicating that even if $\boldsymbol{A}$ is a wide matrix, the large $\mathrm{Ker}(\boldsymbol{A})$ can still miss the small $\mathcal{C}_s$. Consequently, sparse recovery through 1-norm minimization is possible even if m is very small.)

2. (a) Consider the random walk of a particle along the real line. At each time step of size $\tau > 0$, the particle jumps left or right with a distance $h > 0$ with equal probability $1/2$. Let x and t denote the space and time variables, respectively. Derive an equation for the probability density of the particle at (x,t), as the time/space steps $\tau, h \searrow 0$ in the limit $h^2/\tau \to d$.

   (b) Consider a particle moving randomly in a 2-dimensional space, where it can move up/down/left/right with a distance $h > 0$ with equal probability $p \le \frac{1}{4}$. Let (x,y) and t denote the space and time variables, respectively. Derive an equation for the probability density of the particle at (x,y,t), as the time/space steps $\tau, h \searrow 0$ in the limit $h^2/\tau \to d$.

(c) Consider the problem in item (2) over a finite time interval $(0,T)$ with $T > 0$ signifying the terminal time. Let $u(\mathbf{z},t)$, $\mathbf{z} = (x,y)$, denote the probability density and let p and d be both positive constants. Assume the initial distribution of u is given by $f(\mathbf{z})$. Derive the system to model the particle's motion in terms of u. We further assumed that f is compactly supported, i.e. there is a bounded domain $\Omega \Subset \mathbb{R}^2$ such that $\mathrm{supp}(f) \Subset \Omega$. Let us consider the problem of finding the initial distribution f by monitoring the motion of the particle on the boundary $\partial\Omega$. To that end, we introduce

$$\mathcal{M}_f = u(\mathbf{z},t)|_{(\mathbf{z},t)\in\partial\Omega\times(0,T)}.$$

(c-i). Can we uniquely determine f by knowledge of $\mathcal{M}_f$? That is, is the correspondence between f and $\mathcal{M}_f$ one-to-one?

(c-ii). Suppose that f is a delta distribution of the form $\alpha_0\delta(\mathbf{z}-\mathbf{z_0})$, with $\mathbf{z_0} \in \Omega$ and $\alpha_0 \in \mathbb{R}_+$. Here, δ is the Kronecker delta function. Can one determine the initial distribution f by knowledge of $\mathcal{M}_f$? If so, can you describe a numerical scheme of locating $\mathbf{z_0}$, independent of α_0?

(c-iii). If f is a collection of sparsely distributed point distributions, namely,

$$f(x) = \sum_{j=1}^{N} \alpha_j \delta(\mathbf{z}-\mathbf{z_j}),\ \ \alpha_j \in \mathbb{R},\ \ \mathbf{z_j} \in \Omega,$$

$|\mathbf{z_j} - \mathbf{z_{j'}}| \gg 1$, $1 \le j \ne j' \le N$, can you sketch an idea of extending your result in item (b)?